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TRANSITION BIFURCATION BRANCHES IN NON-LINEAR WATER WAVES

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SUMMARY

We are concerned with the numerical computation of progressive free surface gravity waves on a horizontal bed. They are regarded as families of bifurcation branches $(\lambda, A)_Q$ of constant discharge Q.

Numerically we determine two transition values Q_1 and Q_2 with corresponding transition bifurcation branches that classify waves into three disjoint branch sets B_1, B_2 and B_3. Their members are families of waves $(\lambda, A)_Q$ satisfying the conditions $0 < Q^2 \leqslant Q_1^2, Q_1^2 < Q^2 \leqslant Q_2^2$ and $Q_2^2 < Q^2 < B/27$, respectively.

The bifurcation patterns are analysed in some detail from the computed bifurcation diagram, which shows that in B_1 bifurcation is to the left and the amplitude A increases as the wavelength λ decreases; in B_2 bifurcation is to the right and turning points are observed nearly at breaking point. In B_3 bifurcation is to the right and A increases monotonically with λ.

KEY WORDS Water Waves Bifurcation Non-linear

INTRODUCTION

The subject of numerical computation of progressive free surface gravity waves has made significant progress in recent years.[1–4] The wide variety of numerical techniques used includes perturbation expansions, boundary integral methods, finite difference, finite-element and boundary element methods.

In most formulations of the water wave problems the undisturbed water depth is taken as an independent parameter. Results are thus presented as families of waves of constant depth. Also it is usual to perform computations in terms of non-dimensional wavelength ($\lambda = 2\pi$). All this is well suited for boundary integral and perturbation expansion techniques which have proved very accurate for the problem of waves on a horizontal bed.

Variational techniques,[5,6] in conjunction with the finite element method,[7] the Kantorovich method[8] etc., have proved efficient in the computation of problems of engineering interest in which the bed profile may not be uniform. Examples of such problems are critical flows over weirs,[7,9] flows over spillway crests,[10,11] waves created by upstream obstructions,[12,13] etc. In the variational formulation of these free surface problems the discharge Q and the wavelength λ (or domain length) are independent parameters. Their relative behaviour, their interdependent ranges of physical significance, the question of multiple solutions, etc. are problems of theoretical as well as of practical computational importance.

In this paper we study numerically the Q–λ relationship for the case of non-linear water waves on a horizontal bed. The work is aimed at determining two transition values Q_1 and Q_2 of the discharge that completely determine three disjoint wave regions B_1, B_2 and B_3.

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The problem is interpreted as a bifurcation problem in which we compute branches of solutions $(\lambda, A)_Q$ for constant Q bifurcating from the uniform solution of zero amplitude A. The transition branches corresponding to Q_1 and Q_2 are the boundaries between B_1 and B_2 and B_2 and B_3, respectively.

The bifurcation points (λ_1, D_1) and (λ_2, D_2) of the transition branches give two transition wavelengths (λ_1 and λ_2) and two transition asymptotic depths (D_1 and D_2).

The regions B_1, B_2 and B_3 are determined by the Q^2-ranges $(0, Q_1^2]$, $(Q_1^2, Q_2^2]$ and $(Q_2^2, 8/27)$, respectively. In B_1 bifurcation is to the left and A increases as λ decreases. In B_2 bifurcation is to the right and a right turning point is observed which gives rise to multiple solutions. In B_3 bifurcation is to the right and A is observed to increase monotonically with λ.

The numerical computations are carried out using a new Kantorovich algorithm, the details of which are reported elsewhere.[8]

COMPUTATIONAL DETAILS

The progressive free surface gravity waves considered in this paper are assumed to be two-dimensional, irrotational, steady, incompressible, non-viscous and with no surface tension. These non-linear waves are governed by a variational principle[5] with functional

$$J_{Q,\lambda}[h(x), \psi(x,y)] = \int_0^\lambda \int_{-1}^{-1+h(x)} [\tfrac{1}{2}(\nabla\psi)^2 - y]\,dx\,dy \tag{1}$$

and the constraints

$$\psi = 0 \text{ on the bed } y = -1 \text{ and } \psi = Q \text{ on the free surface.} \tag{2}$$

The position of the free surface $y = -1 + h(x)$ is governed by $h(x)$ and the internal flow field distribution is given by the volumetric stream function $\psi(x, y)$. The parameters Q and λ (the discharge and domain length, respectively) are prescribed whereas the unknowns $h(x)$ and $\psi(x, y)$ arise as the result of computation.

All quantities in (1) and (2) have been non-dimensionalized with respect to length H_0 (the total head or stagnation level) and time $(H_0/g)^{1/2}$, where g denotes the acceleration due to gravity.

Boundary conditions at the inlet and outlet boundaries (which are made to coincide with a crest or trough) are those of normal flow and arise as natural conditions in the variational formulation.

The computations reported on in this paper were carried out using a Kantorovich method[8] based on (1) and (2). The technique consists of assuming an expansion for J in which ψ is expressed as some series of functions in y with coefficients which are functions of x. Truncation of the series after N terms and the stationary conditions give a system of N non-linear ordinary differential equations with appropriate boundary conditions. Numerical solution of this boundary value problem gives the position of N streamlines including that of the free surface. The algorithm is also applicable to other free surface problems with arbitrary bed profile $b(x)$, and full details of the implementation are given in Reference 8.

For a prescribed value of the discharge Q with $0 < Q^2 < 8/27$ and a value of λ in an appropriate subinterval of $(0, \infty)$ a wave of amplitude A may be computed. In Figure 1, as an example, we show a full computed wave for $Q^2 = 0{\cdot}2691909$ and $\lambda = 4{\cdot}25$. The computed amplitude is 0·207578.

It is well known that for a value of Q in the given range there are two asymptotic solutions D_r and D_0 which are the positive roots of the cubic

$$2h^3 - 2h^2 + Q^2 = 0. \tag{3}$$

D_r is termed the rapid (or supercritical) solution and D_0 the tranquil (or subcritical) solution.

In the context of the present paper we shall call D_0 the trivial solution and it will be denoted by (λ, D_0). The trivial solution can be computed for any (positive) value of the domain length λ. Non-trivial solutions ($A > 0$) will be points on a branch $(\lambda, A)_Q$ bifurcating at a point (λ_0, D_0) as illustrated in Figure 2.

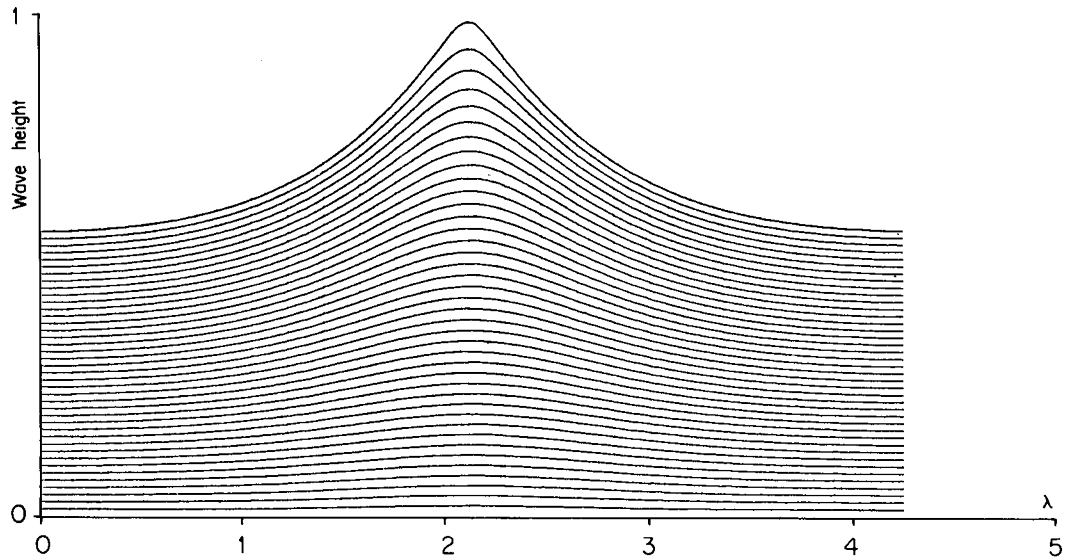

Figure 1. Computed wave of amplitude $A = 0{\cdot}207578$ for prescribed $Q^2 = 0{\cdot}2691909$ and $\lambda = 4{\cdot}25$

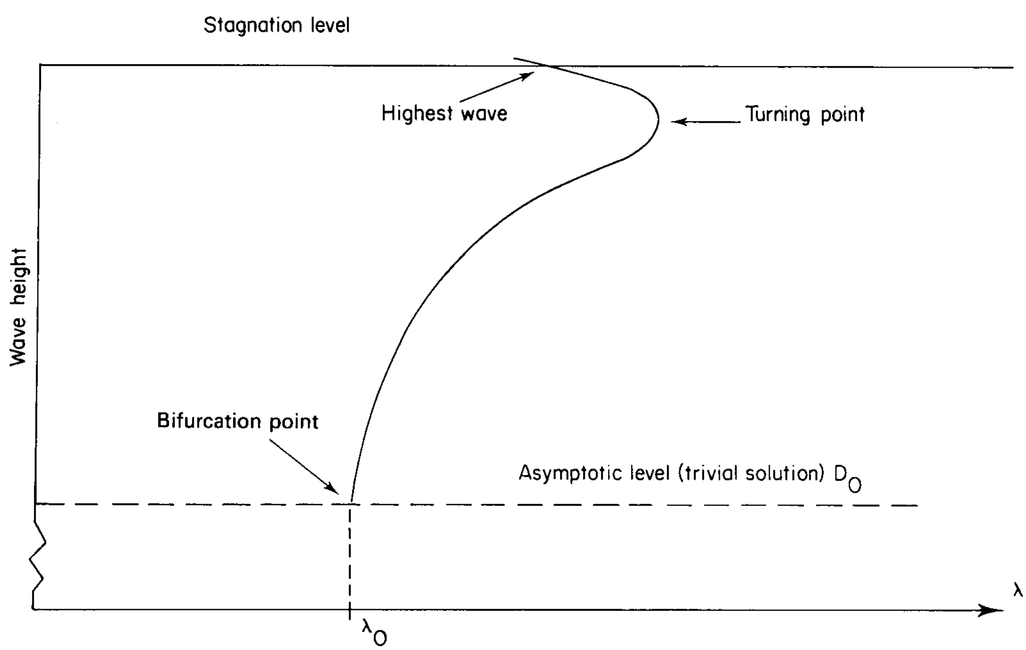

Figure 2. Sketched bifurcation branch $(\lambda, A)_Q$ from trivial solution at $\lambda = \lambda_0$

The bifurcation point satisfies the linear wave theory relation

$$\lambda_0 = 4\pi(1 - D_0)/\tanh(2\pi D_0/\lambda_0). \tag{4}$$

The full bifurcation branch $(\lambda, A)_Q$ is determined by computing a number of points (complete wave solutions) on it, typically twenty. It should be remarked that the computation of a single points on a branch implies a certain computational effort. For instance for the computed wave of Figure 1 we used a 40 × 120 mesh and thus solved 4800 algebraic (non-linear) equations.

Some of the questions arising are: (i) Is bifurcation to the left or to the right? (ii) If the bifurcation pattern changes where does it occur? (iii) Are there turning points? etc.

The main theme of this paper is the numerical computation of two transition values Q_1 and Q_2 of the discharge that determine two transition branches $(\lambda, A)_{Q_1}$ and $(\lambda, A)_{Q_2}$. These branches separate three wave regions B_1, B_2 and B_3 whose bifurcation patterns answer the questions posed above.

DETERMINATION OF THE TRANSITION BRANCHES

Computed bifurcation branches $(\lambda, A)_Q$ for many values of the discharge Q give a bifurcation diagram as illustrated in Figure 3. There, we have chosen three representative curves b_i of the three branch regions B_i suggested by the diagram.

Since each branch has constant discharge Q we seek a definition of B_1, B_2 and B_3 in terms of Q. Hence the boundary branches will be determined by two values Q_1 and Q_2, the computation of which is best handled in terms of the corresponding bifurcation points (λ_1, D_1) and (λ_2, D_2) as illustrated in Figure 4. As discussed previously the sought-for discharge values Q_1 and Q_2 are related to D_1 and D_2 via equation (3) and these are related to λ_1 and λ_2 via equation (4).

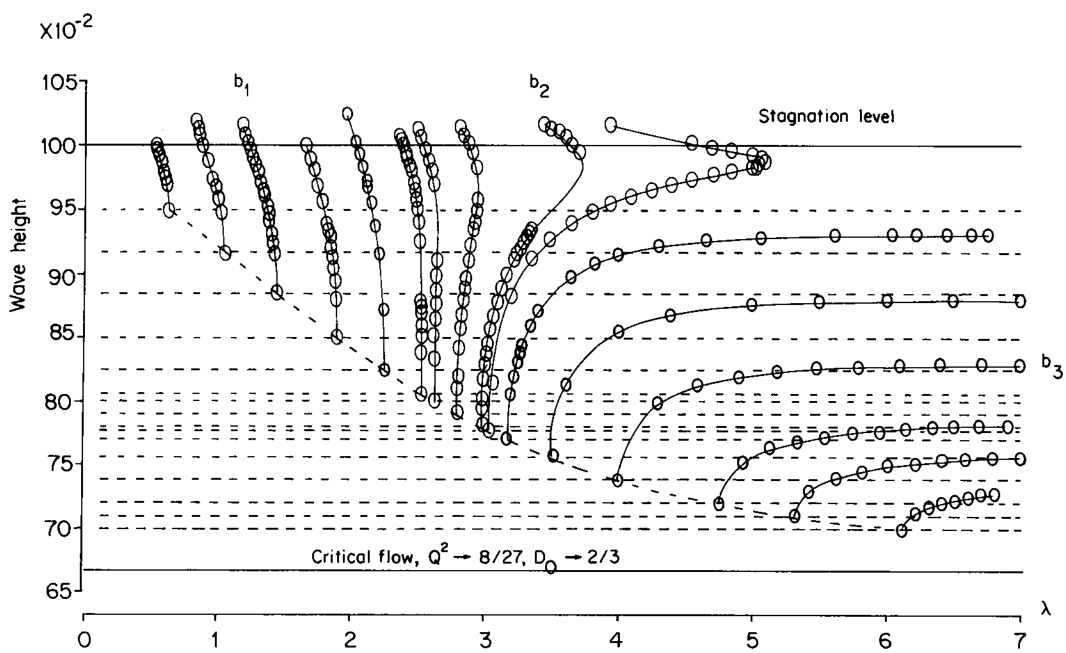

Figure 3. Computed bifurcation diagram

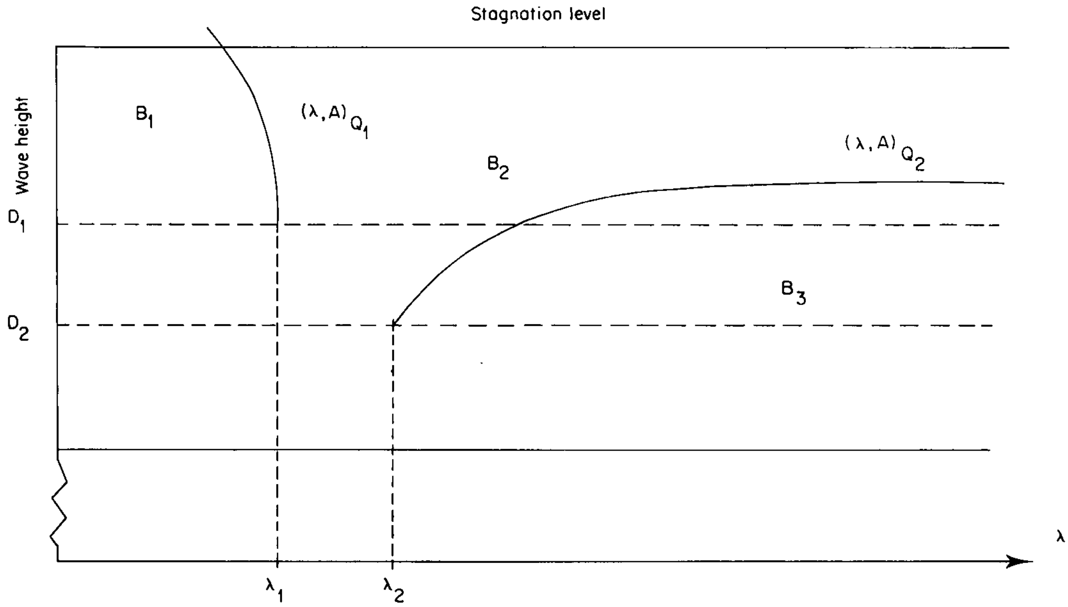

Figure 4. Sketch of transition bifurcation branches defining B_1, B_2 and B_3.

Computation of λ_1

We begin by considering several computed bifurcation branches $(\lambda, A)_{Q^i}$ for several guessed values Q^i of the discharge. We seek the λ-co-ordinate of the bifurcation point of a branch that is the right boundary of the region B_1 of all branches bifurcating to the left (see Figures 3 and 4). From the computed results illustrated in the diagram of Figure 3 this value λ_1 appears to be about 2·5. By considering the inverse curves $\lambda^i(A)$, Figure 3 also indicates that the derivative $d\lambda/dA$ should change from negative to positive for a finite range $(0, \varepsilon)$ of A, as we pass from B_1 to B_2.

The computation of λ_1 proceeds iteratively, as illustrated by the flow chart of Figure 5. For each computed curve $\lambda^i(A)$ we consider its derivative $d\lambda^i/dA$ at a fixed amplitude value $A = A_0$ which is chosen arbitrarily (e.g. $A_0 = 0{\cdot}011$). Since each curve is defined by a discrete set of points, the numerical calculation of d^i necessitates the use of curve fitting and interpolation procedures for which the NAG routines E01BAF, E02BBF and E02BCF are used.

For each number d^i there is a wavelength λ_0^i associated with it which is the λ-co-ordinate of the bifurcation point of the branch $(\lambda, A)_{Q^i}$. Interpolation to $d = 0$ from the set of points (d^i, λ_0^i) gives a new value λ_0^{i+1} in the iteration procedure. Use of this new value in equation (4) gives a new asymptotic depth D_0^{i+1} and this gives a new discharge value Q^{i+1} from equation (3). A new bifurcation branch $(\lambda, A)_{Q^{i+1}}$ can now be computed.

The iteration procedure is stopped if the derivative d^{i+1} is less than a preassigned small value TOL (typically 10^{-5}) and the sought solution is taken as $\lambda_1 = \lambda_0^{i+1}$. When computing a new branch we need only consider moderately large values of the amplitude A to account for the local curve behaviour. Six points per curve are found to be sufficient.

The computed solution for the transition wavelength λ_1 is 2·53142. In Table I we give numerical values of points (d^i, λ_0^i) used in the iteration procedure for a chosen value of A_0. Also displayed there are the corresponding discharge values (squared) and the derivative values d^i of each curve at $A = A_0$.

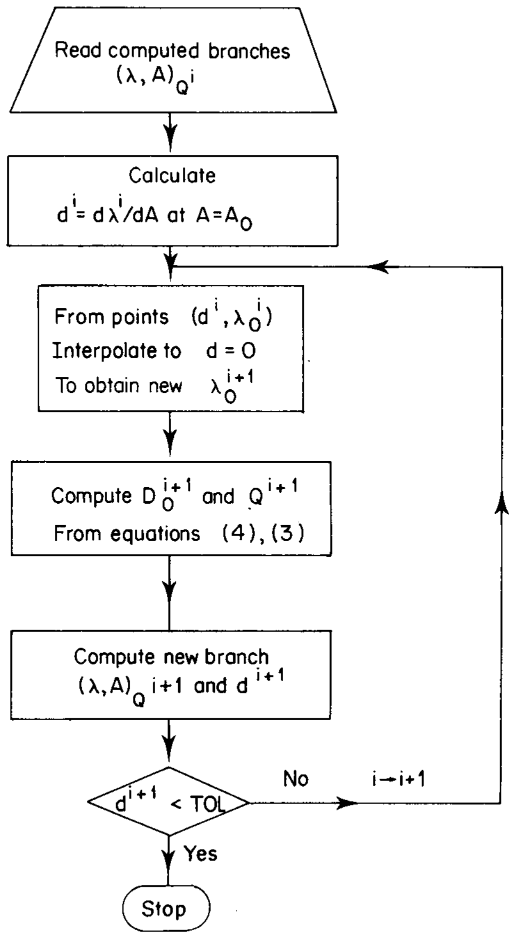

Figure 5. Flow chart for the computation of λ_1.

Table I. Numerical values in iteration procedure for finding transition wavelength $\lambda_1(A_0 = 0{\cdot}01151)$

Curve	$(Q^i)^2$	D_0^i	λ_0^i	d^i
1	0·1400463	0·9166667	1·04723	−0·62399
2	0·2386050	0·8245062	2·24987	−0·12384
3	0·2497685	0·8093515	2·47586	−0·02898
4	0·2521845	0·8058132	2·53124	−0·00046
5	0·2522635	0·8056958	2·53309	0·00240
6	0·2529761	0·8046307	2·55000	0·00829
7	0·2560000	0·8000000	2·62482	0·03225
8	0·2691909	0·7771403	3·03392	0·31845

By noting that λ_1/D_0 is approximately π we speculate that λ_1 has (from equation (4)) the exact value

$$\lambda_1 = 4\pi/(4 + \tanh(2)) = 2{\cdot}5314865\ldots \quad (5)$$

This is only a conjecture which remains to be rigorously checked through analytical methods. However, numerical evidence supports expression (5) for λ_1. Smaller values of the fixed amplitude level A_0 have the effect of increasing the computed value of λ_1 and thus making it closer to the suggested exact value. At the same time it is reasonable to suppose that it is the smaller values of A_0 which would give the more reliable results. In fact direct substitution of λ_1 from (5) gives a solution branch $(\lambda, A)_Q$ that has all the features of the sought-for transition branch to seven decimal places. Therefore we take as the computed solution $\lambda_1 = 2{\cdot}5314856$. From equations (4) and (3) we obtain $D_0 = 0{\cdot}8057973$ and $Q_1^2 = 0{\cdot}2521952$, respectively.

We have found a transition value Q_1 of the discharge such that all bifurcation branches $(\lambda, A)_Q$ with $Q \leqslant Q_1$ bifurcate to the left, A increases as λ decreases and a highest wave exists as the intersection point of a branch with the stagnation level. These branches are the members of the family B_1.

Determination of λ_2

Computationally, as illustrated by the bifurcation diagram of Figure 3, we have found a family of branches that bifurcate to the right and have turning points before intersecting the stagnation level. Also, we observe another family whose members bifurcate to the right and tend to a maximum value below the stagnation level.

Theoretically, it is known[14,15] that conoidal and shallow water waves may be interpreted as families of waves whose amplitudes are bounded above by that of a solitary wave. This can be seen by analysing the phase-plane picture of the Korteweg–de Vries equation for instance.

One may therefore think of the transition branch $(\lambda, A)_{Q_2}$ as that associated with the highest solitary wave, i.e. the solitary wave of discharge Q_2. Since we do not know the detailed behaviour of the branches $(\lambda, A)_Q$ as $\lambda \to \infty$, or equivalently as $Q^2 \to 8/27$, there is a degree of speculation here. As before the determination of Q_2 is carried out via the bifurcation point (λ_2, D_2) of the corresponding bifurcation branch (see Figure 4).

Several numerical solutions for the limiting solitary wave have been published. Williams[3] gives the solution $H/D = 0{\cdot}833197$ where H is the wave height above the (supercritical) asymptotic level D. More recently Hunter and Vanden-Broeck[16] have given the solution $H/D = 0{\cdot}83322$.

Assuming Williams's solution, in the units of this paper, we have $(1 - D)/D = 0{\cdot}833197$, i.e. $D = 0{\cdot}545495$. Recalling that D is the smallest positive solution D_r of the cubic (3) we obtain $Q^2 = 0{\cdot}2704895$ and therefore the tranquil asymptotic level D_0 is $0{\cdot}7745861$. Substitution of D_0 into equation (4) gives $\lambda_0 = 3{\cdot}0848677$. Hence the sought-for bifurcation point is $\lambda_2 = 3{\cdot}0848677$, $D_2 = 0{\cdot}7745861$ and the transition discharge value is $Q_2 = (0{\cdot}2704895)^{1/2}$.

We have thus determined the discharge values Q_1 and Q_2 of the transition bifurcation branches $(\lambda, A)_{Q_1}$ and $(\lambda, A)_{Q_2}$ together with their bifurcation points (λ_1, D_1) and (λ_2, D_2) (see Figure 4).

CONCLUDING REMARKS

Two transition values Q_1 and Q_2 of the discharge Q have been computed which classify all water waves in terms of the bifurcation patterns of three families B_1, B_2 and B_3. Two corresponding transition branches $(\lambda, A)_{Q_1}$ and $(\lambda, A)_{Q_2}$ separate B_1 from B_2 and B_2 from B_3, respectively.

The calculated bifurcation points (λ_1, D_1) and (λ_2, D_2) of these two branches may be interpreted as giving two transition wavelengths λ_1 and λ_2 or two transition asymptotic depths D_1 and D_2.

From numerical evidence the salient features of B_1, B_2 and B_3 have also been indicated. Branches in B_1 satisfy $0 < Q^2 \leqslant Q_1^2$ and $0 < \lambda \leqslant \lambda_1 = 2{\cdot}5314856$; they bifurcation to the left and A increases as λ decreases. B_1 may be regarded as a deep-water wave region and λ_1 as the transition wavelength into this region.

Branches $(\lambda, A)_Q$ in B_2 satisfy $Q_1^2 < Q^2 \leqslant Q_2^2$; bifurcation is to the right and for the Q-cases computed there is a turning point before intersecting the stagnation level. However, we are not certain whether the turning point persists as a feature of each branch in B_2 as $Q^2 \to Q_2^2 = 0{\cdot}2704895$, i.e. for large values of λ.

Branches $(\lambda, A)_Q$ in B_3 satisfy $Q_2^2 < Q^2 < 8/27$ and $\lambda > \lambda_2 = 3{\cdot}0848677$. Bifurcation is to the right and A increases monotonically with λ. Unlike B_1 and B_2 members of B_3 do not have a highest wave.

Although there is still uncertainty about the detailed behaviour of branches in B_2 and B_3 as $\lambda \to \infty$ the results of the present paper may effectively be used in practical applications, e.g. critical flows over weirs, waves due to the presence of upstream obstructions etc.

The results are indeed useful when computing non-linear waves regardless of the method. For instance highest waves can only be expected for $Q^2 \leqslant Q_2^2$. Also, the features of B_2 give a clear warning regarding the computation of the highest waves. It would be tempting to extrapolate from computed points below the turning point.

The computed bifurcation diagram of Figure 3 also provides information as to which wave zones are bound to cause computational difficulties, e.g. near λ_1 and near every turning point. It is also apparent that the parameter λ loses significance at large values.

Finally, although B_1 may be identified with a deep-water wave region, B_2 and B_3 do no strictly represent an intermediate and a shallow-water wave region, respectively, in the classical sense.

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